Economics 611 Handout #1

Game theory - a general theory of strategic interaction
Games of strategy, not games of skill - Equally intelligent players

Game form: players / rules (e.g., admissibility of strategies) / outcomes

Game: Game form plus preferences

Games: Cooperative vs. noncooperative

Simultaneous-Move Games with Perfect Information

In pure strategies 

\[ \text{game form: } (I, \{S_i\}, g(\cdot)) \]

\[ \text{game } \Gamma = (I, \{S_i\}, g(\cdot), \{u_i\}) \text{ or } (I, \{S_i\}, \{u_i(g(\cdot))\}) \]

The game form is common knowledge.

Common knowledge:

If everyone knows x, then x is known up to level 0.
If everyone knows x, and (everyone knows that everyone knows x), then x is known up to level 1.
If everyone knows x, and (everyone knows that everyone knows x), and (everyone knows that (everyone knows that everyone knows x)), then x is known up to level 2.

....

Then x is common knowledge if it is known up to level t for all t ≥ 0.

Solution Concept: What we might reasonably expect to observe.

Profile: \( s = (s_1, s_2, ..., s_n) \) with \( s_j \in S_j \) for all j

\[ s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n) \]

\[ S_{-i} = \{(s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n) : s_j \in S_j \text{ for all } j \neq i\} \]
Strategy $s_i'$ **strictly dominates** strategy $s_i$ in game $\Gamma = (I, \{S_i\}, \{u_i\})$ if

for all $s_i$ in $S_i$, $u_i(s_i', s_i) > u_i(s_i, s_i)$.

Strategy $s_i$ is *strictly dominated* if there exists a strategy $s_i'$ that strictly dominates $s_i$.

Strategy $s_i$ is *strictly dominant* if for all $s_i' \neq s_i$, $s_i$ strictly dominates $s_i'$.

"If a player has a strictly dominant strategy, ... we should expect him to play it. ... It is compelling that players should play strictly dominant strategies if they have them."

Profile $(s_1, s_2, ..., s_n)$ is a *strictly dominant strategy equilibrium* if each $s_i$ is strictly dominant in $S_i$.

Strictly dominant strategy equilibria need not exist.

In outcome space $X$, $x$ is *Pareto superior* to $y$, if $x R_i y$ for all $i$, and $x P_i y$ for at least one $i$. $x$ is *Pareto optimal* in $X$ if there is no $x^*$ in $x$ such that $x^*$ is Pareto superior to $x$. [Notice this is NOT the same as saying that $x$ is Pareto superior to every $x^*$ in $X$ such that $x^* \neq x$.]

**Important!!** It is possible that there is a strictly dominant strategy equilibrium $(s_1, s_2, ..., s_n)$ and yet $g(s_1, s_2, ..., s_n)$ is NOT Pareto optimal, i.e., $g(s_1, s_2, ..., s_n)$ is Pareto dominated by some other feasible alternative. **Prisoner’s dilemma.**
Nash equilibrium

In pure strategies:

First approach: Player i's **best response correspondence**, \( b_i : S_i \to S_i \) in game \((\mathcal{I}, \{S_i\}, \{u_i(\cdot)\})\)

is defined by \( b_i(s_i) = \{s_i' \in S_i : u_i(s_i', s_i) \geq u_i(s_i', s_i) \text{ for all } s_i' \in S_i \} \). Then \((s_1, s_2, \ldots, s_n)\) is a **Nash equilibrium** of game \((\mathcal{I}, \{S_i\}, \{u_i(\cdot)\})\) iff \( s_i \in b_i(s_i) \) for all \( i \in \mathcal{I} \).

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Second approach: i-variant profile: A strategy profile \((s_1', s_2', \ldots, s_n')\) is an **i-variant** of strategy profile \((s_1, s_2, \ldots, s_n)\) if \( s_j' = s_j \) for all \( j \neq i \)

A strategy profile \((s_1, s_2, \ldots, s_n)\) is a **Nash equilibrium** if there does not exist an i and an i-variant profile \((s_1', s_2', \ldots, s_n')\) with: \( u_i(g(s_1', s_2', \ldots, s_n')) > u_i(g(s_1, s_2, \ldots, s_n)) \).

The Cournot Game

Two firms (1,2): duopoly

Demand: \( P = a - b(Q_1 + Q_2) \)

[Actually if we were being very careful, we’d preclude negative prices and say:

\[ P = a - b(Q_1 + Q_2) \text{ if } Q_1 + Q_2 < a/b \text{ and } P = 0 \text{ if } Q_1 + Q_2 \geq a/b. \] ]

\[ \pi_i = P \cdot Q_i - cQ_i \quad \text{for } i = 1, 2. \]

Here we only treat the economically interesting case: \( a, b, \text{ and } c \) are all positive and \( a > c \).

\[ \pi_i = \left[a - b(Q_1 + Q_2)\right]Q_i - cQ_i \]

For a best response function we want to find the global maximizing value of \( Q_i \) for the function
\( \pi_i(Q_1) = [a - b(Q_1 + Q_2)] \cdot Q_1 - cQ_1 = aQ_1 - bQ_1^2 - bQ_2Q_1 - cQ_1 = (a - bQ_2 - c)Q_1 - bQ_1^2 \)

over the domain \( \mathbb{R}_+ \), the non-negative real numbers.

**Existence of a global maximum:** Note that \( \pi_i(Q_1) < 0 \) if

\[
(a - bQ_2 - c)Q_1 - bQ_1^2 < 0 \\
(a - bQ_2 - c)Q_1 < bQ_1^2 \\
(a - bQ_2 - c) < bQ_1 \quad \text{if} \ Q_1 \neq 0 \\
(a - bQ_2 - c)/b < Q_1
\]

So consider the subdomain of \( \pi_i(Q_1) \) that is the interval \( D = [0, (a - bQ_2 - c)/b] \). This is a compact (closed and bounded) set on which \( \pi_i(Q_1) = [a - b(Q_1 + Q_2)] \cdot Q_1 - cQ_1 \) is a continuous function. By the Weierstrass theorem (Extreme value theorem), there exists a \( Q_1^* \) in this interval that maximizes \( \pi_i(Q_1) \) over the interval, i.e., \( \pi_i(Q_1^*) \geq \pi_i(Q_1) \) for all \( Q_1 \in [0, (a - bQ_2 - c)/b] \). In particular, \( \pi_i(Q_1^*) \geq \pi_i(0) = 0 \). Since \( \pi_i(Q_1) < 0 \) for all \( Q_1 > (a - bQ_2 - c)/b \), we see that \( Q_1^* \) maximizes \( \pi_i(Q_1) \) not just over the interval, but over all of \( \mathbb{R}_+ \), i.e., \( Q_1^* \) is a global maximizing value.

**Candidates:** In general, candidates for a global maximum must be either

(1) Boundary points of the domain \( \mathbb{R}_+ \), i.e., \( Q_1 = 0 \); 

(2) Interior points of the domain where \( \pi_i(Q_1) \) fails to be differentiable - and there are none here; or

(3) Interior points of the domain where the function is differentiable and the derivative takes on the value 0. We turn to this category:

\[ D_{Q_1} \pi_i = (a - bQ_2 - c) - 2bQ_1 \]

\[ (a - bQ_2 - c) - 2bQ_1^* = 0. \] So there is a unique critical point:
\[ Q_1 = \frac{a - bQ_2 - c}{2b} \]

**Best response function:** If \( (a - bQ_2 - c) \leq 0 \), i.e., \( Q_2 \geq (a - c)/b \), then we have just a single candidate, \( Q_1 = 0 \), which must then be the global maximum. If \( (a - bQ_2 - c) > 0 \), i.e., \( Q_2 < (a - c)/b \), then we have two candidates, \( Q_1 = 0 \), and \( Q_1 = (a - bQ_2 - c)/2b \), and we must compare \( \pi_i(Q_1) \) at those two values. Of course, \( \pi_i(0) = 0 \). What is \( \pi_i((a - bQ_2 - c)/2b) \)? Since \( \pi_i(Q_1) = (a - bQ_2 - c)Q_1 - bQ_1^2 \),

\[
\pi_i((a - bQ_2 - c)/2b) = (a - bQ_2 - c)(a - bQ_2 - c)/2b - b[(a - bQ_2 - c)/2b]^2, \\
= (a - bQ_2 - c)^2[(1/2b) - (1/4b)] = (a - bQ_2 - c)^2(1/4b),
\]

which is positive since \( (a - bQ_2 - c) > 0 \). Therefore \( \pi_i((a - bQ_2 - c)/2b) > \pi_i(0) \).

Putting all this together, firm #1’s best response function is

\[ Q_1 = \frac{a - bQ_2 - c}{2b} \text{ if } a - c - bQ_2 > 0; \text{ and } 0 \text{ otherwise.} \]

Similarly for firm #2:

\[ Q_2 = \frac{a - bQ_1 - c}{2b} \text{ if } a - c - bQ_1 > 0; \text{ and } 0 \text{ otherwise.} \]
Nash equilibria:

These two candidates for each of the two players lead to just four apparently possible Nash equilibria:

1. Simultaneous solutions of $Q_1 = (a - bQ_2 - c)/2b$ and $Q_2 = 0$: $((a - c)/2b, 0)$
2. Simultaneous solutions of $Q_1 = 0$ and $Q_2 = (a - bQ_2 - c)/2b$: $(0, (a - c)/2b)$
3. Simultaneous solutions of $Q_1 = 0$ and $Q_2 = 0$: $(0, 0)$
4. Simultaneous solutions of $Q_1 = (a - bQ_2 - c)/2b$ and $Q_2 = (a - bQ_2 - c)/2b$: $((a - c)/3b, (a - c)/3b)$

[Do the algebra to get this.]

But none of the first three are Nash equilibria. Consider the first, where #2 chooses 0. As we’ve seen, that is #2's best response just when $(a - bQ_1 - c) \geq 0$. But if $Q_1 = (a - c)/2b,$

$$a - bQ_1 - c = a - b[(a - c)/2b] - c = (a - c) - (a - c)/2 = (a - c)/2 > 0.$$ 

So #2's 0 is NOT a best response to #1's $(a - c)/2b$. Similarly, (2) and (3) can be eliminated.

What remains is the strategy profile $((a - c)/3b, (a - c)/3b)$, but we have to check that, too: Is #1's strategy $(a - c)/3b$ a best response to #2's strategy $(a - c)/3b$? As we have seen, the answer is “Yes” if and only if $(a - bQ_2 - c) > 0$, i.e., if $Q_2 < (a - c)/b$. But obviously, $(a - c)/3b$ is less than $(a - c)/b$, so #1's strategy $(a - c)/3b$ is a best response to #2's strategy $(a - c)/3b$ and #2's strategy $(a - c)/3b$ is a best response to #1's strategy $(a - c)/3b$, i.e., case (4) IS a Nash equilibrium.
Common knowledge / The game of red hats

Once upon a time, many years ago, in a Kingdom far, far away, a cruel King, who was known by all to always tell the truth, decided to have sport with the n prisoners in his castle jail. The prisoners were all brought to the circular Great Hall where they were placed, evenly spread out with their backs to the wall. Blindfolded, each had a hat placed on their head. Before the blindfolds were removed, each was told that if they looked up at their own hat or if they tried to communicate with any other prisoner, they would be immediately beheaded.

“All of the hats are either blue or red,” the King said. “We are going to play a game in a sequence of steps. At the first step, a gong will sound and each of you has the chance to raise your hand or not. If you actually have a red hat and you raise your hand, you immediately will be set free. If you have a blue hat and you raise your hand, you will be immediately beheaded. Then a bell will ring to signal the end of the first step. Then a gong will signal the start of the second round and the same rules apply: If you actually have a red hat and you raise your hand, you immediately will be set free. If you have a blue hat and you raise your hand, you will be immediately beheaded. Then a bell will signal the end of the second step. This will continue indefinitely.”

What happens depends on (1) what the situation is (i.e., how many are wearing red hats), (2) what is known about the situation, and (3) the level of commonality of what is known about the situation.

Case 1. Exactly one hat is red.

   Subcase 1A. That at least one hat is red is known up to level 0.
   Subcase 1B. That at least one hat is red is known up to level 1.
   Subcase 1C. That at least one hat is red is known up to level 2.

Case 2. Exactly two hats are red.

   Subcase 2A. That at least one hat is red is known up to level 0.
   Subcase 2B. That at least one hat is red is known up to level 1.
   Subcase 2C. That at least one hat is red is known up to level 2.
   Subcase 2D. That at least two hats are red is known up to level 0.
   Subcase 2E. That at least two hats are red is known up to level 1.
Case 3. Exactly three hats are red.

Subcase 3A. That at least one hat is red is known up to level 0.
Subcase 3B. That at least one hat is red is known up to level 1.
Subcase 3C. That at least one hat is red is known up to level 2.
Subcase 2D. That at least two hats are red is known up to level 0.
Subcase 2E. That at least two hats are red is known up to level 1.
Subcase 2F. That at least three hats are red is known up to level 0.
Subcase 2G. That at least three hats are red is known up to level 1.

Exercises:

1. There are two players. Player 1’s strategy space is \{0,1,2,...,20\} (numbers of nickels offered to #2); player 2’s strategy space is all 21-entry lists of A (accept) and R (reject). The payoff at (x, arrarr...ar) is
   a) 100-5x for #1 and 5x for #2 if #2 has a in the x+1st position
   b) (0,0) otherwise.
What are all the Nash equilibria?

2. A. (Autarky) Firm A in the US is the only seller of wine there. Let \(Q^A_{US}\) be Firm A’s sales in the US where demand is given by \(6 - \frac{1}{8}Q^A_{US}\). Firm A’s costs are 1 per unit.

\[
\pi^A(Q^A_{US}) = (6 - \frac{1}{8}Q^A_{US}) \cdot Q^A_{US} - Q^A_{US}
\]

What value of \(Q^A_{US}\) maximizes profit? What is the sum of consumer surplus and firm profit?

B. (Free Trade) Now trade is opened between the US and France. Firm A is still the only producer in the US, while firm B is the only producer in France. Both can sell in either country: \(Q^A_{US}\) is A’s sales in the US; \(Q^B_{Fr}\) is A’s sales in France; \(Q^A_{US}\) is B’s sales in the US; \(Q^B_{Fr}\) is B’s sales in France. If a firm sells in the other country, their costs include not only their production costs but also a transportation cost of 1 per unit. Demand in the US is \(6 - \frac{1}{8}(Q^A_{US} + Q^B_{US})\) while demand in France is \(6 - \frac{1}{8}(Q^A_{Fr} + Q^B_{Fr})\). There is an important separability here; e.g.,
\[ \pi^A(Q^A_{US}, Q^A_{Fr}, Q^B_{US}, Q^B_{Fr}) = \]

\[ P_{US}(Q^A_{US} + Q^B_{US}) - Q^A_{US} + P_{Fr}(Q^A_{Fr} + Q^B_{Fr}) - Q^A_{Fr} - (Q^A_{US} + Q^A_{Fr}) - Q^A_{Fr} \]

\[ = [6 - (1/8)(Q^A_{US} + Q^B_{US})]Q^A_{US} + [6 - (1/8)(Q^A_{Fr} + Q^B_{Fr})]Q^A_{Fr} - Q^A_{US} - 2Q^A_{Fr} \]

\[ = \{[6 - (1/8)(Q^A_{US} + Q^B_{US})]Q^A_{US} - Q^A_{US}\} + \{[6 - (1/8)(Q^A_{Fr} + Q^B_{Fr})]Q^A_{Fr} - 2Q^A_{Fr}\} \]

\[ = \pi^A_1(Q^A_{US}, Q^B_{US}) + \pi^A_2(Q^A_{Fr}, Q^B_{Fr}) \]

Assume A and B act as Cournot duopolists in each of the US and French markets. What are the Nash equilibrium values of \( Q^A_{US}, Q^A_{Fr}, Q^B_{US}, \) and \( Q^B_{Fr} \)? What is the value of the sum (US consumer surplus + Firm A’s profit)?

3. (One-sided tariff) Now assume the US imposes on Firm B a tariff of 0.75 on each unit Firm B exports to the US. France has no subsidy and no tariff. What are the Nash equilibrium values of \( Q^A_{US}, Q^A_{Fr}, Q^B_{US}, \) and \( Q^B_{Fr} \)? What is the value of the sum (US consumer surplus + Firm A’s profit + US tariff revenues)? [Assume the same demand curves, same production costs and same transportation costs as in the previous problem. Also assume A and B act as Cournot duopolists in both the US and Fr markets.]

4. Analyze the red hat question for Case 3.

5. Player 1 has 5 strategies and player 2 has 6 strategies. Are there payoffs such that there are no Nash equilibria? Are there payoffs such that there are 30 Nash equilibria? Are there payoffs such that there are 17 Nash equilibria? What numbers of Nash equilibria are possible? Generalize to k strategies for player 1 and ℓ strategies for player 2.