Extensive Form Games

Example 1. Player 1 offers player 2 a number, \( x \), of nickels from one to 20; Player 2 says “accept” or “reject”. If reject, both get 0; if accept, 2 receives 5x and player 1 receives 100-5x.

Example 2. (Bierman and Fernandez) Professor Brown announces he is going to auction off a dollar. Bids proceed in increments of 50 cents. Bidders can not bid twice in a row, and once a bidder passes she does not get to bid again. The highest bidder gets the dollar, but both the highest and second-highest bidders pay their bids to him. Mary and Tom are the only two bidders; it is common knowledge that they each have only $2 in their wallets; and Mary gets to make the first bid.

Example 3. Entry deterrence (MWG, Fig 9.B.1)

<table>
<thead>
<tr>
<th></th>
<th>Fight if E plays “In”</th>
<th>Accommodate if E plays “In”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>In</td>
<td>-3,-1</td>
<td>2,1</td>
</tr>
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</table>
Information sets:  (MWG, Fig 9.C.1 - - Selten’s horse)

Finite game form in extensive form (with perfect recall):

An 11-tuple $GF = (X, Y, A, I, p(\cdot), a(\cdot), H, H(\cdot), t(\cdot), p(\cdot), g(\cdot))$

$X$: a finite set of nodes

$Y$: a space of outcomes

$A$: a finite set of possible actions

$I$: a finite set of players  $I = \{1, 2, ..., I\}$  \{+ “0”, Nature\}

$p(\cdot)$: the immediate predecessor function.  $p(\cdot): X \rightarrow X \cup \{\varnothing\}$

$p(x) = \varnothing$ for exactly one $x \in X$, called the initial node and labeled $x_0$

$s(x)$: the immediate successor correspondence.  $s(x) = p^{-1}(x)$. 
T : the set of terminal nodes \[ T = \{ x \in X : s(x) = \emptyset \} \]

\( X \setminus T \) : the decision nodes

\( P(x) \) : the set of all predecessors of \( x \) : given \( y \in X \), \( y \in P(x) \) if and only if there exists a sequence (a "path") \( x_1, x_2, \ldots, x_m \) such that \( x_1 = x \); \( x_m = y \); for all \( i \), \( 2 \leq i \leq m \), \( x_i = p(x_{i-1}) \). Such a path is unique.

\( S(x) \) : the set of all successors of \( x \) : given \( y \in X \), \( y \in S(x) \) if and only if there exists a sequence (a "path") \( x_1, x_2, \ldots, x_m \) such that \( x_1 = x \); \( x_m = y \); for all \( i \), \( 2 \leq i \leq m \), \( x_i \in s(x_{i-1}) \). Such a path is unique.

We require, for all \( x \in X \), that \( P(x) \cap S(x) = \emptyset \)

\( \alpha(\cdot) \) : action function \( \alpha(\cdot) : X \setminus \{x_0\} \to A \)

\( \alpha(x) \) is an action leading from \( p(x) \) to \( x \).

If \( y \in P(x) \), there is a unique path \( x_1, x_2, \ldots, x_m \) such that \( x_1 = x \); \( x_m = y \); for all \( i \), \( 2 \leq i \leq m \), \( x_i = p(x_{i-1}) \). We say \( \alpha(x_{m-1}) \) is the action at \( y \) on the path to \( x \).

We require, for all \( x’ \), \( x'' \in s(x) \) with \( x’ \neq x'' \), that \( \alpha(x’) \neq \alpha(x’’) \).

\( c(x) \) is the choice set at \( x \) : \( c(x) = \{ a \in A : a = \alpha(x’) \text{ for some } x’ \in s(x) \} \)

\( H \) : a collection of information sets that partition \( X \)

\( GF \) is a game form of perfect information if every \( h \in H \) is a singleton

\( H(\cdot) \) : an information function \( H(\cdot) : X \to H \)

We require, for all \( x, x’ \in X \), that \( H(x’) = H(x) \) implies \( c(x’) = c(x) \)

Therefore we can write \( c(h) \) for each \( c(x) \) when \( h = H(x) \)
u(·) : an information assignment function  \( u(·) : H \to \mathcal{I} \cup \{0\} \)

The collection of player \( i \)'s information sets:

\[ H_i = \{ h \in H : u(h) = i \} \]

**Perfect recall:**

(i) If \( H(x') = H(x) \), then \( x' \notin P(x) \cup S(x) \) (and \( x \notin P(x') \cup S(x') \) )

(ii) If \( H(x') = H(x) \), \( x'' \in P(x) \), with \( u(H(x)) = u(H(x'')) \), and \( a \) is the action

at \( x'' \) on the path to \( x \), then there exists an \( x* \in P(x') \cap H(x'') \) such

that the action at \( x* \) on the path to \( x' \) is also \( a \).

p(·) : a probability function (for nature)  \( p(·) : H_0 \times A \to [0, 1] \)

We require  

(i) \( p(h, a) = 0 \) if \( a \notin c(h) \)

(ii) \( \sum_{a \in c(h)} p(h, a) = 1 \) for all \( h \in H_0 \)

\( g(·) : \) an outcome function  \( g(·) : T \to Y \)

**Finite game in extensive form (with perfect recall):** An ordered pair \((GF, u)\) consisting of a

finite game form in extensive form (with perfect recall) together with a collection

\( u = \{ u_1(·), u_2(·), ..., u_i(·) \} \) of utility functions:  \( u_i(·) : Y \to \mathbb{R} \), the reals

We assume each \( u_i(·) \) is a Bernoulli utility function [Expected utility]

**Common knowledge:** Everybody knows the game form, etc.; everybody knows that everybody

knows the game form, etc.; everybody knows that everybody knows that everybody

knows the game form, etc.; ...
A (pure) strategy for player $i \in I$ in game form $GF = (X, Y, A, I, p(\cdot), \alpha(\cdot), H, H(\cdot), u(\cdot), \rho(\cdot), g(\cdot))$ is a function $s_i : H_i \to A$ such that $s_i(h) \in c(h)$ for all $h \in H_i$.

The collection of all of $i$'s strategies is $S_i$.

Mas-Colell, Whinston, and Green, Exercise 7.D.1

A profile of strategies: $s = (s_1, s_2, \ldots, s_i)$ where $s_i \in S_i$ for all $i \in I$.

$s = (s_1, s_2, \ldots, s_i) = (s_i, s_{-i})$

$S = S_1 \times S_2 \times \ldots \times S_n$ and $S_{-i} = S_1 \times S_2 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$

The normal form representation of game form $GF$ is $(I, \{S_i\})$

**General Theme of Chapter 9:** Use dynamic structure (extensive form) to eliminate those NE that result from non-credible actions. [Some NE are not sensible predictions.]

Entry deterrence game (a second look)

**Backward Induction for Games of Perfect Information.**

**Zermelo's Theorem.** For every game of perfect information, there exists at least one profile of strategies that can be obtained by backward induction. Any such profile is a Nash equilibrium. If no player has the same payoff at any two terminal nodes, this equilibrium is unique.

Given an extensive form game $\Gamma = (X, Y, A, I, p(\cdot), \alpha(\cdot), H, H(\cdot), u(\cdot), \rho(\cdot), g(\cdot), u(\cdot))$, we say game $\Gamma^* = (X^*, Y^*, A^*, I^*, p^*(\cdot), \alpha^*(\cdot), H^*, H^*(\cdot), \tau^*(\cdot), \rho^*(\cdot), g^*(\cdot), u^*(\cdot))$ is a subgame of $\Gamma$ if
1. \( X^* \subseteq X \);

2. \( Y^* \subseteq Y \);

3. \( A^* \subseteq A \);

4. \( I^* \subseteq I \);

5. \( p^* : X^* \rightarrow X^* \cup \emptyset \); there exists a unique element \( x^* \) of \( X^* \) such that \( p^*(x^*) = \emptyset \), and for all \( x \) in \( X^* \setminus \{x^*\} \), \( p^*(x) = p(x) \) and \( p^*(x) \in X^* \). Important: We require for each \( x \) in \( X^* \), that \( S^*(x) = S(x) \).

6. \( \alpha^* : X^* \setminus \{x^*\} \rightarrow A^* \); with \( \alpha^*(x) = \alpha(x) \) for all \( x \in X^* \setminus \{x^*\} \);

7. \( X^* \subseteq H \);

8. \( H^* : X^* \rightarrow H^* \); with \( H^*(x) = H(x) \) for all \( x \in X^* \setminus \{x^*\} \) AND \( \#H^*(x) = 1 \);

[So if decision node \( x \) is in the subgame, then every \( x' \) in \( H(x) \) is also.]

9. \( t^* : H^* \rightarrow I^* \cup \{0\} \); with \( t^*(h) = u(h) \) for all \( h \in H^* \);

10. \( \rho^* : H^*_0 \times A^* \rightarrow [0, 1] \); satisfies \( \rho^*(h,a) = \rho(h,a) \) for all \( h \in H^*_0 \) and \( a \in A^* \);

11. \( g^* : T^* \rightarrow Y^* \); and \( g^*(t) = g(t) \) for all \( t \in T^* \);

12. \( u^* : Y^* \rightarrow \mathbb{R} \); with \( u^*(y) = u(y) \) for all \( y \) in \( Y^* \).

Given a mixed strategy profile \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_i) \) in a game \( \Gamma \) and a subgame \( \Gamma^* \) of \( \Gamma \), we say \( \sigma \) induces a Nash equilibrium in \( \Gamma^* \) if the moves specified in \( \sigma \) when applied to information sets in \( \Gamma^* \) constitute a Nash equilibrium for \( \Gamma^* \). A mixed strategy profile \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_i) \) in a game \( \Gamma \) is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in every
The entry deterrence game (a third look)

The Nash equilibria identified by backward induction in finite games of perfect information are SPNE.


"...the SPNE concept insists that players should play an SPNE wherever they find themselves in a game tree, even after a sequence of events that is contrary to the predictions of the theory.... players will assume that the remaining play of the game will be an SPNE even if play up to that point has contradicted the theory."

The subgame approach does not allow us to uncover all non-credible actions.

Example 9.C.1 again

Exercises from old exams:

I. At time 0, an incumbent firm (firm I) is already in the widget market, and a potential entrant firm (firm E) is considering entry. In order to enter, firm E must incur a cost of $K > 0$. Firm E’s only opportunity to enter is at time 0. There are three production periods. In any period in which both firms are active in the market, the game in the figure below is played. Firm E moves first, deciding whether to stay in or exit the market. If it stays in, firm I decides whether to fight. Once firm E plays “out”, it is out of the market forever; firm E earns 0 in any period during which it is out of the market, and firm I earns $x$. The discount factor for both firms is $\delta$; i.e., a dollar gained in period 1 is valued at $1; a dollar earned in period 2 is valued at $\delta$; and a dollar earned in period 3 is valued at $\delta^2$. Similarly for costs.

Assume: 1. $x > z > y$; 2. $y + \delta x > (1 + \delta) z$; 3. $1 + \delta > K$; 4. $0 < \delta < 1$.

A. (15) What are the SPNE of this game?

B. (5) Is there a NE that is not a SPNE?
2. In a variation on the game of problem 1, suppose now that firm E faces a financial constraint. In particular if firm I fights once against firm E (in any period), firm E will be forced out of the market from that point on.

   A. (15) What are the SPNE of this game?

   B. (5) Is there a NE that is not a SPNE?

3. For the following game, are there any subgame perfect Nash equilibria? If so, what are they? Are there any other Nash equilibria? Separately treat the cases \( x \geq 0 \) and \( x < 0 \).
4. There are two players. Player #1 offers a point $s_i \in \mathbb{R}$. Player #2 can then accept $s_i$ or reject it; in the latter case the outcome, $s$, is the status quo value $s_0 \in \mathbb{R}$. If #2 accepts the outcome, $s$, is $s_1$. In both parts below, player #2's preferences are represented by $-(s - b)^2$ where $b$ is #2's bliss point.

(Part 1) In this part, player #1's preferences are represented by $s$ (i.e., she prefers higher values of $s$). Find the subgame perfect Nash equilibria. (Hint: Your answer may depend on the sizes of the parameters $s_0$ and $b_2$).

(Part 2) In this part, player #1's preferences are represented by $-(s - b_1)^2$ where $b_1$ is #1's bliss point. Find the subgame perfect Nash equilibria. (Hint: Your answer may depend on the sizes of the parameters $s_0$, $b_1$, and $b_2$).

5. Consider this variation on the previous question: There are two periods. In the first period the game of Question 4 is played. In the second period, the game is played again, but the default status quo point is now the outcome of the first period game.

Payoffs are the undiscounted utilities of the outcome at the end of the second period.

Are there any subgame perfect Nash equilibria? If so, what are they?

[Assume $s_0 < b_2$.]