Euclidean Space

Vectors (Points): \( x = (x_1, x_2, \ldots, x_n) \) e.g., \( 0 = (0, 0, \ldots, 0) \)

Euclidean space: \( \mathbb{R}^n \)

\( x = y; x \geq y; x > y \) \( [x_2 > x_1 \text{ is not the denial of } x_1 \geq x_2] \)

\( \mathbb{R}^n; \mathbb{R}^n_+ \)

Euclidean Metric

\[ d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \]

\[ d(x, y) \geq 0 \]
\[ d(x, y) = d(y, x) \]
\[ d(x, y) = 0 \text{ if and only if } x = y \]
\[ d(x, y) + d(y, z) \geq d(x, z) \]

Homework: Show \( d(0, x) - d(0, y) \leq d(0, x + y) \leq d(0, x) + d(0, y) \)

Open \( \epsilon \)-ball at \( x \) in \( \mathbb{R}^n \): For \( \epsilon > 0 \), \( B_\epsilon(x) = \{ x^* \in \mathbb{R}^n / d(x^*, x) < \epsilon \} \)

S is bounded if there exists an \( \epsilon > 0 \) such that \( S \subset B_\epsilon(0) \).

Examples

Intersections, Unions, \( A \times B \) (Cartesian product)

Topology on \( \mathbb{R}^n \)

Let S be a set in \( \mathbb{R}^n \):

S is open if for every \( x \) in \( S \), there exists an \( \epsilon > 0 \) such that \( B_\epsilon(x) \subset S \).

Examples

Intersections, Unions, \( A \times B \)

Int(S), the interior of \( S \), is the union of all the open subsets of \( S \).
Int(S) = \{x \in S / \text{there exists an } \epsilon > 0 \text{ with } B_{\epsilon}(x) \subset S \}.

S is \textit{closed} if \( \mathbb{R}^n \setminus S \) is open.

Examples

Intersections, Unions, \( A \times B \)

\textbf{Limit point}: \( x \) is a limit point of \( S \) if every \( \epsilon \)-ball at \( x \) contains a point of \( S \) other than \( x \).

\( S \) is closed iff it contains all its limit points.

The \textit{boundary} of a set \( S \), \( \partial(S) \), is the collection of all points that are both limit points of \( S \) and limit points of the complement of \( S \).

\textbf{Theorem}: If \( g: X \subset \mathbb{R}^n \to \mathbb{R} \) is continuous on closed \( X \), then for any \( c \) in \( \mathbb{R} \), both of the sets \( \{x \in X / g(x) \geq c\} \) and \( \{x \in X / g(x) \leq c\} \) are closed; equivalently, both of the sets \( \{x \in X / g(x) < c\} \) and \( \{x \in X / g(x) > c\} \) are open.

\textbf{Sums of Sets}

Given \( A, B \subset \mathbb{R}^n \): \( A + B = \{z \in \mathbb{R}^n / \text{there exist } x \in A \text{ and } y \in B \text{ such that } z = x + y \} \)

Examples:

1. In \( \mathbb{R}^2 \), if \( A = \{(a, 0) / a \in \mathbb{R}\} \) and \( B = \{(0, b) / b \in \mathbb{R}\} \), then \( A + B = \mathbb{R}^2 \)

2. ![](image)

\textbf{Exercises}:

1. \( B_{\epsilon}(0) + \{x\} = B_{\epsilon}(x) \)

2. \( A + \emptyset = A; A + \mathbb{R}^n = \mathbb{R}^n \)

3. What is the sum of \( A = \{(x,y) / x \leq 0 \text{ and } y = 0\} \) and \( B = \{(x,y) / 0 \leq x \text{ and } -1 + 1/(1 + x^2) \leq y \leq 1 - 1/(1 + x^2)\}? \)
Which of the following are true?

4. $A \neq \emptyset, B \neq \emptyset$ imply $A + B \neq \emptyset$.

5. $A \subseteq A + B$.

6. $\{x\} + B = \{x\} + C$ implies $B = C$.


8. $A$ and $B$ bounded implies $A + B$ is bounded.

9. $f(A + B) = f(A) + f(B)$.

10. For each of the following properties, suppose each $Y^j$ has the property. Does that imply that $\sum Y^j$ must have the property?
   A. Non-empty
   B. Additive: $y, y^* \in Y^j$ implies $y + y^* \in Y^j$
   C. Divisibility and constant returns to scale: $y \in Y^j$ and $\beta > 0$ imply $\beta y \in Y^j$
   D. Possibility of inaction: $0 \in Y^j$
   E. Convexity
   F. No free lunch: $Y^j \cap \mathbb{R}^n_+ \subseteq \{0\}$
   G. Open
   H. Closed

**Convexity**

Set $S$ in $\mathbb{R}^n$ is *convex* if for every $x, y \in S$ and every $\lambda \in [0,1]$,
$$\lambda x + (1 - \lambda)y \in S.$$  

Set $S$ in $\mathbb{R}^n$ is *strictly convex* if for every $x, y \in S$ and every $\lambda \in (0,1)$,
$$\lambda x + (1 - \lambda)y \in \text{Int}(S).$$

**Exercises:**

1. Is $\emptyset$ convex? Strictly convex?

2. Is $\{x\}$ convex? Strictly convex?

3. If $A$ is convex and $B$ is strictly convex, is $A + B$ strictly convex?

4. If $A$, $B$, and $C$ are convex and $A + B = A + C$, does that imply $B = C$?