The First Fundamental Theorem of Welfare Economics: Suppose that $\succ_i$ satisfies local non-satiation for all $i$. If a state

$$z^0 = (x_1^0, ..., x_m^0, y_0^0, ..., y_n^0)$$

is an equilibrium relative to price vector $p$. Then $z^0$ is Pareto optimal.

Proof: (Debreu: “the easiest proof in all of mathematical economics”)

Since $z^0$ is an equilibrium, it is feasible. So if it were not Pareto-optimal, there would exist a Pareto-superior state $z' = (x_1', x_2', ..., x_m', y_1', y_2', ..., y_n')$ that is feasible. We will show that the existence of such a $z'$ leads to a contradiction.

By the feasibility of $z'$, each $x_i' \in X^i$. By Pareto superiority, $x_i' \succ_i x_i$ for all $i$ and $x_i' \succ_i x_i$ for at least one $i$, say $i = 1$. From $x_1' \in X^1$, $x_1' \succ_i x_i$, and $x_i \in \delta_i(p)$ we have

$$p \cdot x_1' > p \cdot x_1.$$  

(1)

From $x_i' \in X^i$, $x_i' \succ_i x_i$, and $x_i \in \delta_i(p)$ we have

$$p \cdot x_i' \geq p \cdot x_i.$$  

(2)

by Consumer Duality Theorem #1. Adding up (1) and (2) for all $i = 2, 3, ..., m$, we get

$$\sum_{i=1}^{m} p \cdot x_i' > \sum_{i=1}^{m} p \cdot x_i,$$

or, equivalently,

$$\sum_{i=1}^{m} p \cdot x_i' - \sum_{i=1}^{m} p \cdot x_i > 0$$

On the production side, since $y_i' \in Y^i$ and $y_i \in \varphi^i(p)$, we must have

$$p \cdot y_i \geq p \cdot y_i'.$$

Adding these up:

$$\sum_{j=1}^{n} p \cdot y_j \geq \sum_{j=1}^{n} p \cdot y_j'.$$
or

\[ 0 > \sum_{j=1}^{n} p_{j}y_{j}' - \sum_{j=1}^{n} p_{j}y_{j} \]

Combining the two summation results:

\[ \sum_{i=1}^{m} p_{i}x_{i}' - \sum_{i=1}^{m} p_{i}x_{i} > \sum_{j=1}^{n} p_{j}y_{j}' - \sum_{j=1}^{n} p_{j}y_{j} \]

or, equivalently

\[ \sum_{i=1}^{m} p_{i}x_{i}' - \sum_{j=1}^{n} p_{j}y_{j}' > \sum_{i=1}^{m} p_{i}x_{i} - \sum_{j=1}^{n} p_{j}y_{j} \]

or, finally,

\[ p'[\sum_{i=1}^{m} x_{i}' - \sum_{j=1}^{n} y_{j}'] > p'[\sum_{i=1}^{m} x_{i} - \sum_{j=1}^{n} y_{j}] \]

But, by the materials balance part of feasibility

\[ [\sum_{i=1}^{m} x_{i}' - \sum_{j=1}^{n} y_{j}'] = \sum_{i=1}^{m} \omega_{i} = [\sum_{i=1}^{m} x_{i} - \sum_{j=1}^{n} y_{j}] \]

Combining the last two displays:

\[ p'[\sum_{i=1}^{m} \omega_{i}] > p'[\sum_{i=1}^{m} \omega_{j}] \]

This contradiction has arisen from the assumption that \( z \) is not Pareto-optimal. Hence it is. \( \square \)

Several things need to be said about this First Fundamental Theorem:
Thing 1. The explicit assumptions are very weak - at least in comparison with those made in the Second Theorem. The only assumption about consumers is local non-satiation; and there are no explicit assumptions about producers. In particular, there are no convexity assumptions about $X^i$, $Y^i$, or $\s_i^j$. However, non-convexities - like those caused by increasing returns, may yield a situation in which no competitive equilibrium exists. The First Theorem is uninformative when it can’t be supplemented with a proof of equilibrium existence.

Thing 2. There are strong implicit assumptions. Externalities have been ruled out and, in the presence of externalities, competitive equilibria need not be Pareto optimal. Also, it is assumed that agents will follow the “rules” of the game, but we will see that they have incentives to do otherwise, and that acting on these incentives will yield outcomes that are not Pareto optimal.

Thing 3. “Always-achieving-Pareto-optimality.” has weaknesses as a criterion for resource allocation procedures.

(A) Some Pareto optima are terrible, especially on distributional grounds. A resource allocation procedure that always gave everything to #1 has the same property shown above for competitive markets, but would be considered unfortunate by all (except #1). The criterion of the First Theorem is unappealing in the absence of a corresponding Second Theorem.

(B) Related to (A), there are other criteria: justice, fairness, stability, security, individual freedom, etc. We may want to trade away from perfect achievement always of Pareto optimality if it helps substantially on these other dimensions.

(C) A mechanism which always achieves Pareto optimality guarantees a lot of social conflict – there is no room for tension-reducing moves to Pareto-superior states. Anyone’s getting better off comes at the cost of someone else being worse off.

(D) Only the preferences of living people are taken into account by the market mechanism. But the resulting states affect future generations whose preferences “ought” to be taken into account. This is a kind of externality – across generations. And judging social states only by their impact on people allows us to overlook what may be a fundamentally important “keeping in harmony with Nature.” Homocentrism can result in savaging other species and our common environment [see David Ehrenfeld, The Arrogance of Humanism].

Exercises

1. Use the $\s_i^j$ ordering from the first problem of the previous sets of exercises to build an Edgeworth-Bowley box example of an equilibrium that is not Pareto-optimal when local non-satiation fails.
Hyperplanes: Bounding, Supporting, and Separating

Hyperplane: \( H = \{ x \in \mathbb{R}^n / p \cdot x = v \} \) for \( p \neq 0 \) determines two halfspaces:

\[ H^+ = \{ x \in \mathbb{R}^n / p \cdot x \geq v \} \quad \text{and} \quad H^- = \{ x \in \mathbb{R}^n / p \cdot x \leq v \} \]

A hyperplane \( H \) is bounding for a set \( S \) if \( S \subseteq H^+ \) or \( S \subseteq H^- \). \( H \) is bounding for \( S \) through a point \( u \) if also \( u \in H \).

Exercises:
1. In \( \mathbb{R}^2 (\mathbb{R}^3) \), let \( S = B(\mathbf{0}, \epsilon) \); for what points \( u \) is there a hyperplane through \( u \) bounding for \( S \)? What if \( S = \mathbb{R}^2_+ \) ?

2. Suppose \( S = H^+ \) for some \( H = \{ x \in \mathbb{R}^n / p \cdot x = v \} \), that \( u \) is a point not in \( S \) and that \( G = \{ x \in \mathbb{R}^n / q \cdot x = u \} \) is a hyperplane through \( u \) bounding for \( S \). What relationship must there be between \( p \) and \( q \)?

3. Describe those sets \( S \subseteq \mathbb{R}^2 \) such that for every \( u \in \mathbb{R}^2 \) (both in and out of \( S \)), there is a bounding hyperplane for \( S \) through \( u \).

4. Suppose there is a hyperplane bounding for \( S \) through \( u \) and a hyperplane bounding for \( T \) through \( u \). Must there be a hyperplane bounding for \( S + T \) through \( u + v \)? [Hint: No]

Do this again when \( S \) and \( T \) are known to be convex. Same hint.

A hyperplane \( H \) is supporting for a set \( S \) in \( \mathbb{R}^n \) if

1. \( H \) is bounding for \( S \); and
2. \( H \cap \beta(S) \neq \emptyset \).

Exercises:

5. Find a set \( S \), closed in \( \mathbb{R}^2 \), and a point \( u \in S \) such that any hyperplane bounding for \( S \) through \( u \) must be supporting for \( S \).

6. Can a set \( S \) have a bounding hyperplane but no supporting hyperplane?

Given two sets \( S \) and \( T \) in \( \mathbb{R}^n \), a hyperplane \( H \) separates \( S \) and \( T \) if \( S \subseteq H^+ \) and \( T \subseteq H^- \).

7. Consider, in \( \mathbb{R}^2 \), the sets \( \mathbb{R}^2_+ \) and \( \mathbb{R}^2_- = \{ x / -x \in \mathbb{R}^2_+ \} \). Show that there are infinitely many different hyperplanes separating \( \mathbb{R}^2_+ \) and \( \mathbb{R}^2_- \).

8. Is it possible to have sets \( T \) and \( S \subseteq T \) and a hyperplane \( H \) such that \( H \) separates \( S \) and \( T \)?

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Theorem (Minkowski)
1. If K is a convex set and \(z \notin \text{Int}(K)\), then there exists a hyperplane H bounding for K through z.
2. If K and K* are two non-empty convex sets such that \(\text{Int}(K \cap K^*) = \emptyset\), then there is a hyperplane H separating K and K*.

Exercises:
9. If \(\text{Int}(K \cap K^*) \neq \emptyset\), then there is no hyperplane separating K and K*.

The Second Fundamental Theorem of Welfare Economics: Suppose a state \(z = (x_1^0, x_2^0, ..., x_m^0, y_1^0, y_2^0, ..., y_n^0)\) is Pareto-optimal and
1. Each \(X^i\) is convex;
2. \(\Sigma Y^i\) is non-empty and convex;
3. Each \(\equiv_i\) satisfies local non-satiation;
4. Each set \(\{x / x \succ_i x^0_i\}\) is open in \(X^i\);
5. Each set \(\{x / x \preceq_i x^0_i\}\) is convex;
6. For all \(i\), \(x^0_i \in \text{Int} X^i\).

Then there exists a \(p \neq 0\) such that \(z\) is a competitive equilibrium with respect to \(p\) after a redistribution of the endowment and alteration of the \(\theta_i\) share values.

Proof:
(First Part) For each \(i\), define \(X^i_0 = X^i \cap \{x / x \succeq_i x^0_i\}\). By (1) and (5), \(X^i_0\) is convex. Also \(X^i_0\) is non-empty - it contains \(x^0_i\). Hence \(\Sigma X^i_0\) is non-empty and convex. Next define \(\omega = \Sigma \omega_i\). Since \(\Sigma Y^i\) is non-empty and convex so is \(\Sigma Y^i + \omega\).

(Second Part) The two sets \(\Sigma X^i_0\) and \(\Sigma Y^i + \omega\) are each not empty and convex. Their intersection is non-empty (it contains \(x_1^0 + x_2^0 + ... + x_m^0 = y_1^0 + y_2^0 + ... + y_n^0 + \omega\), but their intersection has empty interior. We put off proof of that last part until the end (in the Fourth Part). Now we apply the separating hyperplane theorem: There is a \(c\) and \(p \neq 0\) such that the hyperplane \(H = \{x / p \cdot x = c\}\) has the two properties
(A) \(\Sigma X^i_0 \subset H^+ = \{x / p \cdot x \geq c\}\);
(B) \(\Sigma Y^i + \omega \subset H^- = \{x / p \cdot x \leq c\}\).

Note that since \(x_1^0 + x_2^0 + ... + x_m^0 = y_1^0 + y_2^0 + ... + y_n^0 + \omega\) is in both these sets it must be in \(H\):
\[p \cdot \Sigma x_i^0 = p \cdot (\Sigma y_i^0 + \omega) = c\]

(Third Part) In this part we show that \(z\) is a competitive equilibrium relative to \(p\). We need to prove three things:
1). \(z\) is feasible. This holds since \(z\) is Pareto-optimal.
2). $y_j^0 \in \varphi^i(p)$. To see this, let $y^j$ be any element of $Y^i$. Then

$$y_1^0 + y_2^0 + \ldots + y_{j-1}^0 + y_j^0 + y_{j+1}^0 + \ldots + y_n^0 + \omega \in \Sigma Y^i + \omega \subset H^-$$

so

$$p \cdot (y_1^0 + y_2^0 + \ldots + y_{j-1}^0 + y_j^0 + y_{j+1}^0 + \ldots + y_n^0 + \omega) \leq c = p \cdot (y_1^0 + y_2^0 + \ldots + y_{j-1}^0 + y_j^0 + y_{j+1}^0 + \ldots + y_n^0 + \omega)$$

which implies

$$p \cdot y^j \leq p \cdot y_j^0.$$

3). $x_i^0 \in \delta^i(p)$. We want to invoke Consumer Duality Theorem #2. First we show $x_i^0$ is an expenditure minimizer. So suppose $x^i$ is any element $X^i$ such that $x^i \succeq_i x_i^0$, i.e., $x^i \in X^i_0$. Then

$$x_1^0 + x_2^0 + \ldots + x_{i-1}^0 + x_i^0 + x_{i+1}^0 + \ldots + x_m^0 \in \Sigma X^i_0 \subset H^+$$

so

$$p \cdot (x_1^0 + x_2^0 + \ldots + x_{i-1}^0 + x_i^0 + x_{i+1}^0 + \ldots + x_m^0) \geq c = p \cdot (x_1^0 + x_2^0 + \ldots + x_{i-1}^0 + x_i^0 + x_{i+1}^0 + \ldots + x_m^0)$$

which implies

$$p \cdot x^i \geq p \cdot x_i^0.$$

But applying Consumer Duality Theorem #2 requires showing two assumptions hold:

First we must show $x_i^0$ doesn’t minimize $p \cdot x$ over $X^i$. That is left as an exercise [Hint: Use $x_i^0 \in \text{Int } X^i$ and $p \neq \emptyset$].

Second, we need to know $p \cdot x_i^0$ just uses up i’s wealth. Since

$$p \cdot \Sigma x_i^0 = p \cdot (\Sigma y_j^0) + p \cdot \omega$$

we can reallocate $\omega$ and choose $\theta_{ij}$ values to ensure this.

(Fourth Part) What remains is to show that the interior of $\Sigma X^i_0 \cap \Sigma Y^j + \omega$ is empty. We proceed to assume it is not empty and seek a contradiction. So let

$$x \in \text{Int} (\Sigma X^i_0 \cap \Sigma Y^j + \omega),$$

i.e., there is an $\epsilon$ such that

$$B_\epsilon(x) \subset \Sigma X^i_0 \cap \Sigma Y^j + \omega$$

Since $B_\epsilon(x) \subset \Sigma X^i_0$, we may write $x = x_1 + x_2 + \ldots + x_m$, as the sum of elements from the $X^i_0$ sets.

We will first show

$$\Sigma B_{\epsilon m}(x_i) \subset B_\epsilon(x).$$
[Then $\Sigma B_{e/m}(x_i) \subseteq \Sigma X_{i0}'$, but that does not mean each $B_{e/m}(x_i) \subseteq X_{i0}'$.]

So let $x' = x_1' + x_2' + ... + x_m'$ be an element of $\Sigma B_{e/m}(x_i)$.

$$|x' - x| = |x_1' + x_2' + ... + x_m' - (x_1 + x_2 + ... + x_m)|$$

$$= |(x_1' - x_1) + (x_2' - x_2) + ... + (x_m' - x_m)|$$

$$\leq |(x_1' - x_1)| + |(x_2' - x_2)| + ... + |(x_m' - x_m)|$$

$$< \epsilon/m + \epsilon/m + ... + \epsilon/m = \epsilon.$$  

Therefore $x' \in B_{e}(x)$.

By local non-satiation, for each $i$, there is an $x_i^* \in B_{e/m}(x_i)$ with

$$x_i^* \in X^i \text{ and } x_i^* \succ_i x_i.$$  

Recall $x_i \in X_{i0}'$, so $x_i \preceq_i x_i^0$. Hence we have $x_i^* \succ_i x_i^0$ and $x_i^* \in X^i$ for each $i$.

Turning to the production side,

$$\Sigma x_i^* \in \Sigma B_{e/m}(x_i) \subseteq B_{e}(x) \subseteq \Sigma Y^j + \omega.$$  

Therefore there exist $y_1^*, ..., y_n^*$ with each $y_j^* \in Y^j$ and

$$\Sigma x_i^* = \Sigma y_j^* + \omega.$$  

Hence $(x_1^*, x_2^*, ..., x_m^*, y_1^*, ..., y_n^*)$ is feasible and Pareto-superior to $(x_1^0, x_2^0, ..., x_m^0, y_1^0, y_2^0, ..., y_n^0)$ contrary to the Pareto optimality of $(x_1^0, x_2^0, ..., x_m^0, y_1^0, y_2^0, ..., y_n^0)$. This contradiction has arisen from assuming there exists an $x$ in the interior of $\Sigma X_{i0}' \cap \Sigma Y^j + \omega$, so that interior must be empty.  □

Several things need to be said about the Second Fundamental Theorem.

**Thing 1.** As with the First, there are important implicit assumptions, most notably, the absence of externalities and the assumption that agents tell the truth.

**Thing 2.** Just because $z = (x_1, x_2, ..., x_m, y_1, ..., y_n)$ is a competitive equilibrium with respect to $p$ does not mean the $x_i$ and $y_j$ will be chosen when agents face prices $p$. Decentralized choices at $p$ need not even satisfy the materials balance constraints. We will say $z$ is sustained at $p$ to reflect the idea that there are no incentives to depart from these positions.
**Thing 3.** The convexity assumptions which make their appearance in the Second (but not the First) theorem are crucial. In the absence of convexity, there may exist Pareto optimal equilibria that can not be sustained as competitive equilibria.

The convexity assumptions play several roles:
1. They rule out indivisibilities
2. For production possibility sets, they rule out increasing returns to scale.
3. For consumer upper contour sets, they represent a taste for – or at least no aversion to – variety. As an example of this, Arrow has a nice explanation of how racial or gender discrimination involve non-convexities of upper contour sets. [“The Theory of Discrimination”, 1973, in Aschenfelter and Rees, editors, Discrimination in Labor Markets.]

**Thing 4.** Note that we have required convexity of \( \Sigma y_j \) but not convexity of each \( Y^i \) separately.

**Exercise:** Construct an example showing that \( A + B \) can be convex even though neither \( A \) nor \( B \) is convex.

**Thing 5.** The need for \( x^0_i \in \text{Int} \ X^i \) is seen in “Arrow’s exceptional case”.

**Thing 6.** For the Pareto-optimal state to be sustained as a competitive equilibrium, operation of markets will generally have to be preceded by a one-time redistribution. There is no guarantee that this can be done without redistributing rights to someone’s endowment of their labor time.

**Thing 7.** No claim is made here that competitive markets make up the only resource allocation process that identifies equilibria and Pareto-optimals.

**Exercises.**

1. In the simple Edgeworth-Bowley box example, show that when both consumers have lexicographical preferences there may be Pareto-optimal states that are not equilibria.

2. Draw an Edgeworth-Bowley box with just a lattice of points. Can you form indifference sets so that there is a Pareto-optimal state that can not be sustained as an equilibrium?

3. Suppose there are two individuals with:

   #1: \( U^1 = x_1y_1 \); \( X_1 = \mathbb{R}^2; \omega_1 = (4,1) \);
   #2: \( U^2 = x_2y_2 \); \( X_2 = \mathbb{R}^2; \omega_2 = (1,4) \).
Then \( p = (1,1) \) is a vector of competitive equilibrium prices with outcome

\[
x_1 = x_2 = y_1 = y_2 = 2.5.
\]

Suppose now \( #1 \) considers concealing an amount \( d \) of his endowment of \( x \) to influence equilibrium prices. At the end, \( #1 \) gets to bring out and consume the \( d \) units of \( x \) as well as the amounts of \( x \) and \( y \) he purchases at the new equilibrium prices. What is \( #1 \)'s optimal choice of \( d \)?

4. There is a single apple producer which takes prices as fixed and has technology given by

\[
A = 20L_{A}^{1/2}.
\]

With every apple produced, a unit of nectar is produced: \( N = A \). There is no market for nectar. There is a single honey producer which takes prices as fixed and has technology given by

\[
H = N + L_{H}.
\]

There are 200 households, each endowed with one unit of labor and 1/200th of the rights to the profits of each producer. Each household has preferences given by

\[
U^i = H^i + A^i.
\]

A. Determine the allocation of labor between \( L_A \) and \( L_H \) at competitive equilibrium prices.

B. Determine the allocation of labor between \( L_A \) and \( L_H \) at a Pareto optimum. Compare your result with that in Part (A).

5. A state \( z = (x_1, x_2, ..., x_m, y_1, ..., y_n) \) is called equitable if no consumer prefers anyone else’s bundle to his own:

\[
x_i \succeq x_j \text{ for all } i, j
\]

In the pure exchange case, is every competitive equilibrium equitable? Is every equitable state able to be sustained as a competitive equilibrium? [Assume non-satiation, continuity, convexity, etc, all hold.]